MONISM IN ARITHMETIC.

IN HIS "Primer of Philosophy," Dr. Paul Carus, the able editor of this magazine, defines monism as a "unitary conception of the world." Similarly, we shall understand by monism in a science the unitary conception of that science. The more a science advances the more does monism dominate it. An example of this is furnished by physics. Whereas formerly physics was made up of wholly isolated branches, like Mechanics, Heat, Optics, Electricity, and so forth, each of which received independent explanations, physics has now donned an almost absolute monistic form, by the reduction of all phenomena to the *motions* of molecules. For example, optical and electrical phenomena, we now know, are caused by the undulatory movements of the ether, and the length of the ether-waves constitutes the sole difference between light and electricity.

Still more distinctly than in physics is the monistic element displayed in pure arithmetic, by which we understand the theory of the combination of two numbers into a third by addition and the direct and indirect operations springing out of addition. Pure arithmetic is a science which has completely attained its goal, and which can prove that it has, exclusively by internal evidence. For it may be shown on the one hand that besides the seven familiar operations of addition, subtraction, multiplication, division, involution, evolution, and the finding of logarithms, no other operations are definable which present anything essentially new; and on the other hand that fresh extensions of the domain of numbers beyond irrational, imaginary, and complex numbers are arithmetically impossible. Arithmetic may be compared to a tree that has completed its growth,

the boughs and branches of which may still increase in size or even give forth fresh sprouts, but whose main trunk has attained its fullest development.

Since arithmetic has arrived at its maturity, the more profound investigation of the nature of numbers and their combinations shows that a unitary conception of arithmetic is not only possible but also necessary. If we logically abide by this unitary conception, we arrive, starting from the notion of counting and the allied notion of addition, at all conceivable operations and at all possible extensions of the notion of number. Although previously expressed by Grassmann, Hankel, E. Schröder, and Kronecker, the author of the present article, in his "System of Arithmetic," Potsdam, 1885, was the first to work out the idea referred to, fully and logically and in a form comprehensible for beginners. This book, which Kronecker in his "Notion of Number," an essay published in Zeller's jubilee work, makes special mention of, is intended for persons proposing to learn arithmetic. As that cannot be the object of the readers of this magazine, whose purpose will rather be the study of the logical construction of the science from some single fundamental principle, the following pages will simply give of the notions and laws of arithmetic what is absolutely necessary for an understanding of its development.

The starting-point of arithmetic is the idea of counting and of number as the result of counting. On this subject, the reader is requested to read the author's article in the last number of *The Monist* (p. 396). It is there shown that the idea of addition springs immediately from the idea of counting. As in counting it is indifferent in what order we count, so in addition it is indifferent, for the sum, or the result of the addition, whether we add the first number to the second or the second to the first. This law, which in the symbolic language of arithmetic, is expressed by the formula

$$a + b = b + a,$$

is called the *commutative law* of addition. Notwithstanding this law, however, it is evidently desirable to distinguish the two quantities which are to be summed, and out of which the sum is produced, by special names. As a fact, the two summands usually are distin-

guished in some way, for example, by saying a is to be increased by b, or b is to be added to a, and so forth. Here, it is plain, a is always something that is to be increased, b the increase. Accordingly it has been proposed to call the number which is regarded in addition as the passive number or the one to be changed, the *augend*, and the other which plays the active part, which accomplishes the change, so to speak, the *increment*. Both words are derived from the Latin and are appropriately chosen. Augend is derived fròm *augere*, to increase, and signifies that which is to be increased; increment comes from *increscrere*, to grow, and signifies as in its ordinary meaning what is added.

Besides the commutative law one other follows from the idea of counting—the *associative law* of addition. This law, which has reference not to two but to three numbers, states that having a certain sum, $a+b$, it is indifferent for the result whether we increase the increment b of that sum by a number, or whether we increase the sum itself by the same number. Expressed in the symbolic language of arithmetic this law reads,

$$a+(b+c)=(a+b)+c.$$

To obtain now all the rules of addition we have only to apply the two laws of commutation and association above stated, though frequently, in the deduction of the same rule, each must be applied many times. I may pass over here both the rules and their establishment.

In addition, two numbers, the augend a and the increment b are combined into a third number c, the sum. From this operation spring necessarily two inverse operations, the common feature of which is, that the sum sought in addition is regarded in both as known, and he difference that in the one the augend also is regarded as known, and in the other the increment. If we ask what number added to a gives c, we seek the increment. If we ask what number increased by b gives c, we seek the augend. As a matter of reckoning, the solution of the two questions is the same, since by the commutative law of addition $a+b=b+a$. Consequently, only one common name is in use for the two inverses of addition, namely, *subtraction*. But with respect to the notions involved, the two oper-

ations do differ, and it is accordingly desirable in a logical investigation of the structure of arithmetic, to distinguish the two by different names. As in all probability no terms have yet been suggested for these two kinds of subtraction, I propose here for the first time the following words for the two operations, namely, *detraction* to denote the finding of the increment, and *subtertraction* to denote the finding of the augend. We obtain these terms simply enough by thinking of the augmentation of some object already existing. For example, the cathedral at Cologne had in its tower an augend that waited centuries for its increment, which was only supplied a few decades ago. As the cathedral had originally a height of one hundred and thirty metres, but after completion was increased in height twenty-six metres, of the total height of one hundred and fifty-six metres one hundred and thirty metres is clearly the augend and twenty-six metres the increment. If, now, we wished to recover the augend we should have to pull down (Latin, *detrahere*) the upper part along the whole height. Accordingly, the finding of the augend is called *detraction.* If we sought the increment, we should have to pull out the original part from beneath (Latin, *subtertrahere*). For this reason, the finding of the increment is called *subtertraction.* Owing to the commutative law, the two inverse operations, as matters of computation, fall into one, which bears the name of *subtraction.* The sign of this operation is the minus sign, a horizontal stroke. The number which originally was sum, is called in subtraction minuend; the number which in addition was increment is now called detractor; the number which in addition was augend is now called subtertractor. Comprising the two conceptually different operations in one single operation, subtraction, we employ for the number which before was increment or augend, the term subtrahend, a word which on account of its passive ending is not very good, and for which, accordingly, E. Schröder proposes to substitute the word *subtrahent,* having an active ending. The result of subtraction, or what is the same thing, the number sought, is called the *difference.* The definition-formula of subtraction reads

$$a - b + b = a,$$

that is, a minus b is the number which increased by b gives a, or

the number which added to *b* gives *a*, according as the one or the other of the two operations inverse to addition is meant. From the formula for substraction, and from the rules which hold for addition, follow now at once the rules which refer to both addition and subtraction. These rules we here omit.

From the foregoing it is plain that the minuend is necessarily larger than the subtrahent. For in the process of addition the minuend was the sum, and the sum grew out of the union of *two* natural number-pictures.* Thus 5 minus 9, or 11 minus 12, or 8 minus 8, are combinations of numbers *wholly destitute of meaning*; for no number, that is, no result of counting, exists that added to 9 gives the sum 5, or added to 12 gives the sum 11, or added to 8 gives 8. What, then, is to be done? Shall we banish entirely from arithmetic such meaningless unions of numbers; or, since they have no meaning, shall we rather invest them with one? If we do the first, arithmetic will still stick in the strait-jacket in which it is forced by the original definition of number as the result of counting. If we adopt the latter alternative we are forced to extend our notion of number. But in doing this, we sow the first seeds of the science of pure arithmetic, an organic body of knowledge which fructifies all other provinces of science.

What significance, then, shall we impart to the symbol

$$5 - 9?$$

Since 5 minus 9 possesses no significance whatever, we may, of course, impart to it any significance we may wish. But as a matter of practical convenience it should be invested with no meaning that is likely to render it subject to exceptions. As the form of the symbol $5-9$ is the form of a difference, it will be obviously convenient to give it a meaning which will allow us to reckon with it as we reckon with every other real difference, that is, with a difference in which the minuend is larger than the subtrahent. This being agreed upon, it follows at once that all such symbols in which the number before the minus sign is less than the number behind it by the same amount may be put equal to one another. It is practical,

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* See the article "Notion and Definition of Number" in the last *Monist*.

therefore, to comprise all these symbols under some one single symbol, and to construct this latter symbol so that it will appear unequivocally from it by how much the number before the minus sign is less than the number behind it. This difference, accordingly, is written down and the minus sign placed before it.

If the two numbers of such a differential *form* are equal, a totally new sign must be invented for the expression of the fact, having no relation to the signs which state results of counting. This invention was not made by the ancient Greeks, as one might naturally suppose from the high mathematical attainments of that people, but by Hindu Brahman priests at the end of the fourth century after Christ. The symbol which they invented they called *tsiphra,* empty, whence is derived the English *cipher.* The form of this sign has been different in different times and with different peoples. But for the last two or three centuries, since the symbolic language of arithmetic has become thoroughly established as an international character, the form of the sign has been 0 (French *zéro,* German *null*).

In calling this symbol and the symbols formed of a minus sign followed by a result of counting, *numbers,* we widen the province of numbers, which before was wholly limited to results of counting. In no other way can zero and the negative numbers be introduced into arithmetic. No man can *prove* that 7 minus 11 is equal to 1 minus 5. Originally, both are meaningless symbols. And not until we agree to impart to them a significance which allows us to reckon with them as we reckon with real differences are we led to a statement of identity between 7 minus 11 and 1 minus 5. It was a long time before the negative numbers mentioned acquired the full rights of citizenship in arithmetic. Cardan called them, in his *Ars Magna,* 1545, *numeri ficti* (imaginary numbers), as distinguished from *numeri veri* (real numbers). Not until Descartes, in the first half of the seventeenth century, was any one bold enough to substitute *numeri ficti* and *numeri veri* indiscriminately for the same letter of algebraic expressions.

We have invested thus combinations of signs originally meaningless, in which a smaller number stood before than after a minus sign, with a meaning which enables us to reckon with such *apparent*

differences exactly as we do with ordinary differences. Now it is just this practical shift of imparting meanings to combinations, which logically applied deduces naturally the whole system of arithmetic from the idea of counting and of addition, and which we may characterise, therefore, as the *foundation-principle* of its whole construction. This principle, which Hankel once called the *principle of permanence,* but which I prefer to call the PRINCIPLE OF NO EXCEPTION, may be stated in general terms as follows:

In the construction of arithmetic every combination of two previously defined numbers by a sign for a previously defined operation (plus, minus, times, etc.) shall be invested with meaning, even where the original definition of the operation used excludes such a combination; and the meaning imparted is to be such that the combination considered shall obey the same formula of definition as a combination having from the outset a signification, so that the old laws of reckoning shall still hold good and may still be applied to it.

A person who is competent to apply this principle rigorously and logically will arrive at combinations of numbers whose results are termed irrational or imaginary with the same necessity and facility as at the combinations above discussed, whose results are termed negative numbers and zero. To think of such combinations as *results* and to call the products reached also "numbers" is a misuse of language. It were better if we used the phrase *forms of numbers* for all numbers that are not the results of counting. But *usus tyrannus!*

It will now be my task to show how all numbers at which arithmetic ever has arrived or ever can arrive naturally flow from the simple application of the principle of no exception.

Owing to the commutative and associative laws for addition it is wholly indifferent for the result of a series of additive processes in what order the numbers to be summed are added. For example,

$$a + (b + c + d) + (e + f) = (a + b + c) + (d + e) + f.$$

The necessary consequence of this is that we may neglect the consideration of the order of the numbers and give heed only to what the quantities are that are to be summed, and, when they are equal, take note of only two things, namely, of what the quantity which is

to be repeatedly summed is called and how often it occurs. We thus reach the notion of multiplication. To multiply a by b means to form the sum of b numbers each of which is called a. The number conceived summed is called the multiplicand, the number which indicates or counts how often the first is conceived summed is called the multiplier.

It appears hence, that the multiplier must be a result of counting, or a number in the original sense of the word, but that the multiplicand may be any number hitherto defined, that is, may also be zero or negative. It also follows from this definition that though the multiplicand may be a concrete number the multiplier cannot. Therefore, the commutative law of multiplication does not hold when the multiplicand is concrete. For, to take an example, though there is sense in requiring four trees to be summed three times, there is no sense in conceiving the number three summed "four trees times." When, however, multiplicand and multiplier are unnamed results of counting, (abstract numbers,) two fundamental laws hold in multiplication, exactly analogous to the fundamental laws of addition, namely, the law of commutation and the law of association. Thus,

$$a \textit{ times } b = b \textit{ times } a,$$

$$\text{and, } a \textit{ times } (b \textit{ times } c) = (a \textit{ times } b) \textit{ times } c.$$

The truth and correctness of these laws will be evident, if keeping to the definition of multiplication as an abbreviated addition of equal summands, we go back to the laws of addition. Owing to the commutative law it is unnecessary, for purposes of practical reckoning, to distinguish multiplicand and multiplier. Both have, therefore, a common name: *factor*. The result of the multiplication is called the product; the symbol of multiplication is a dot (.) or a cross ($\times$), which is read "times." Joined with the fundamental formula above written are a group of subsidiary formulæ which give directions how a sum or difference is multiplied and how multiplication is performed with a sum or difference. I need not enter, however, into any discussion of these rules here.

As the combination of two numbers by a sign of multiplication has no significance according to our definition of multiplication,

when the multiplier is zero or a negative number, it will be seen that we are again in a position where it is necessary to apply the above explained principle of no exception. We revert, therefore, to what we above established, that zero and negative numbers are symbols which have the form of differences, and lay down the rule that multiplications with zero and negative numbers shall be performed exactly as with real differences. Why, then, is minus one times minus one, for example, equal to plus one? For no other reason than that minus one can be multiplied with an ordinary difference, as, for example, 8 minus 5, by first multiplying by 8, then multiplying by 5, and subtracting the differences obtained, and because agreeably to the principle of no exception we must say that the multiplication must be performed according to exactly the same rule with a symbol which has the *form* of a difference whose minuend is less by one than its subtrahent.

As from addition two inverse operations, detraction and subtertraction, spring, so also from multiplication two inverse operations must proceed which differ from each other simply in the respect that in the one the multiplicand is sought and in the other the multiplier. As matters of computation, these two inverse operations again meet in a single operation, namely, division, owing to the validity of the commutative law in multiplication. But in so far as they are different ideas, they must be distinguished. As most civilised languages distinguish the two inverse processes of multiplication in the case in which the multiplicand is a line, we will adopt for arithmetic a name which is used in this exception. Let us take this example,

$$4\,\textit{yards} \times 3 = 12\,\textit{yards}.$$

If twelve yards and four yards are given, and the multiplier 3 is sought, I ask, how many summands, each equal to four yards, give twelve yards, or, what is the same thing, how often I can lay a length of four yards on a length of twelve yards. But this is *measuring*. Secondly, if twelve yards and the number 3 are given, and the multiplicand four yards is sought, I ask what summand it is which taken three times gives twelve yards, or, what is the same thing, what part I shall obtain if I cut up twelve yards into three equal parts. But this is partition, or *parting*. If, therefore, the multi-

plier is sought we call the division *measuring*, and if the multiplicand is sought, we call it *parting*. In both cases the number which was originally the product is called the dividend. and the result the quotient. The number which originally was multiplicand is called the measure; the number which originally was multiplier is called the parter. The common name for measure and parter is divisor. The common symbol for both kinds of division is a colon, a horizontal stroke, or a combination of both. Its definitional formula reads,

$$(a \div b) \,.\, b = a, \text{ or, } \frac{a}{b} \,.\, b = a.$$

Accordingly, dividing a by b means, to find the number which multiplied by b gives a, or to find the number *with* which b must be multiplied to produce a. From this formula, together with the formulæ relative to multiplication, the well-known rules of division are derived, which I here pass over.

In the dividend of a quotient only such numbers can have a place which are the product of the divisor with some previously defined number. For example, if the divisor is 5 the dividend can only be 5, 10, 15, and so forth, and 0, -5, -10 and so forth. Accordingly, a stroke of division having underneath it 5 and above it a number different from the numbers just named is a combination of symbols having no meaning. For example, $\frac{3}{5}$ or $\frac{12}{5}$ are meaningless symbols. Now, conformably to the principle of no exception we must invest such symbols which have the form of a quotient without their dividend being the product of the divisor with any number yet defined, with a meaning such that we shall be able to reckon with such apparent quotients as with ordinary quotients. This is done by our agreeing always to put the product of such a quotient form with its divisor equal to its dividend. In this way we reach the definition of broken numbers or *fractions*, which by the application of the principle of no exception spring from division exactly as zero and negative numbers sprang from subtraction. The latter had their origin in the impossibility of the subtraction; the former have their origin in the impossibility of the division. Putting

together now both these extensions of the domain of numbers, we arrive at *negative fractional numbers.*

We pass over the easily deduced rules of computation for fractions and shall only direct the reader's attention to the connexion which exists between fractional and non-fractional or, as we usually say, whole numbers. Since the number 12 lies between the numbers 10 and 15, or, what is the same thing, $10 < 12 < 15$, and since $10:5=2$, $15:5=3$, we say also that $12:5$ lies between 2 and 3, or that

$$2 < \tfrac{12}{5} < 3.$$

In itself, the notion of "less than" has significance only for results of counting. Consequently, it must first be stated what is meant when it is said that 2 is less than $\frac{12}{5}$. Plainly, nothing is meant by this except that 2 times 5 is less than 12. We thus see that every broken number can be so interpolated between two whole numbers differing from each other only by 1 that the one shall be smaller and the other greater, where smaller and greater have the meaning above given.

From the above definitions and the laws of commutation and association all possible rules of computation follow, which in virtue of the principle of no exception now hold indiscriminately for all numbers hitherto defined. It is a consequence of these rules, again, that the combination of two such numbers by means of any of the operations defined must in every case lead to a number which has been already defined, that is, to a positive or negative whole or fractional number, or to zero. The sole exception is the case where such a number is to be divided by zero. If the dividend also is zero, that is, if we have the combination $\frac{0}{0}$, the expression is one which stands for any number whatsoever, because any number whatsoever, no matter what it is, if multiplied by zero gives zero. But if the dividend is not zero but some other number a, be it what it will, we get a quotient form to which *no* number hitherto defined can be equated. But we discover that if we apply the ordinary arithmetical rules to $a \div 0$ all such forms may be equated to one another both when a is positive and also when a is negative. We may therefore invent two new signs for such quotient forms, namely $+\infty$ and

$-\infty$. We find, further, that in transferring the notions greater and less to these symbols, $+\infty$ is greater than any positive number, however great, and $-\infty$ is smaller than any negative number, however small. We read these new signs, accordingly, "plus infinitely great" and "minus infinitely great."

But even here arithmetic has not reached its completion, although the combination of as many previously defined numbers as we please by as many previously defined operations as we please will still lead necessarily to some previously defined number. Every science must make every possible advance, and still one step in advance is possible in arithmetic. For in virtue of the laws of commutation and association, which also fortunately obtain in multiplication, just as we advance from addition to multiplication, so here again we may ascend from multiplication to *an operation of the third degree.* For, just as for $a+a+a+a$ we read $4.a$, so with the same reason we may introduce some more abbreviated designation for $a.a.a.a$. The introduction of this new operation is in itself simply a matter of convenience and not an extension of the ideas of arithmetic. But if after having introduced this operation we repeatedly apply the monistic principle of arithmetic, the principle of no exception, we reach new means of computation which have led to undreamt of advances not only in the hands of mathematicians but also in the hands of natural scientists. The abbreviated designation mentioned, which, fructified by the principle of no exception, can render science such incalculable services, is simply that of writing for a product of b factors of which each is called a, a^b, which we read a to the b^{th} power. Here a new direct operation, that of *involution,* is defined, and from now on we are justified in distinguishing operations which are not inverses of others, as addition, multiplication, and involution, by *numbers of degree.* Addition is the direct operation of the first degree, multiplication that of the second degree, and involution that of the third degree. In the expression a^b the passive number a is called the *base,* the active number b the *exponent,* the result, the *power.*

But whilst in the direct operations of the first and second degree, the laws of commutation and association hold, here in involu-

tion, the operation of third degree, the two laws are inapplicable, and the result of their inapplicability is that operations of a still higher degree than the third form no possible advancement of pure arithmetic. The product of b factors a is not equal to the product of a factors b; that is, the law of commutation does not hold. The only two different integers for which a to the b^{th} power is equal to b to the a^{th} power are 2 and 4, for 2 to the 4^{th} power is 16, and 4 to the second power also is 16. So, too, the law of association as a general rule does not hold. For it is hardly the same thing whether we take the $(b^c)^{th}$ power of a or the c^{th} power of a^b.

From the definition of involution follow the usual rules for reckoning with powers, of which we shall only mention one, namely, that the $(b-c)^{th}$ power of a is equal to the result of the division of a to the b^{th} power by a to the c^{th} power. If we put here c equal to b, we are obliged, by the principle of no exception, to put a to the 0^{th} power equal to 1; a new result not contained in the original notion of involution, for that implied necessarily that the exponent should be a result of counting. Again, if we make b smaller than c we obtain a *negative exponent*, which we should not know how to dispose of if we did not follow our monistic law of arithmetic. According to the latter, a to the $(b-c)^{th}$ power must still remain equal to a^b divided by a^c even when b is smaller than c. Whence follows that a to the minus d^{th} power is equal to 1 divided by a to the d^{th} power, or to take specific numbers, that 3 to the minus 2^{nd} power is equal to $\frac{1}{9}$.

At this point, perhaps, the reader will inquire what a raised to a fractional power is. But this can be explained only when we have discussed the inverse processes of involution, to which we now pass.

If $a^b=c$, we may ask two questions: first, what the base is which raised to the b^{th} power gives c; the second, what the exponent of the power is to which a must be raised to produce c. In the first case we seek the base, and term the operation which yields this result *evolution*; in the second case we seek the exponent and call the operation which yields this exponent, the *finding of the logarithm*. In the first case, we write $\sqrt[b]{c}=a$ (which we read, the b^{th} root of c is equal to a), and call c the *radicand*, b the *exponent of the root*, and a

the *root*. In the second case, we write $\log_a c = b$ (which we read, the logarithm of c to the base a is equal to b), and call c the *logarithmana* or *number*, a the *base of the logarithm*, and b the *logarithm*.

While, owing to the validity of the law of commutation in addition and multiplication, the two inverse processes of those operations are identical so far as computation is concerned, here in the case of involution the two inverse operations are in this regard essentially different, for in this case the law of commutation does not hold.

From the definitional formulæ for evolution and the finding of logarithms, namely,

$$(\sqrt[b]{c})^b = c, \text{ and } (a)^{\log_a c} = c,$$

follow, by the application of the laws of involution, the rules for computation with roots and logarithms. These rules we pass over here, only remarking, first, that for the present $\sqrt[b]{c}$ has meaning only when c is the b^{th} power of some number already defined; and, secondly, that for the present also $\log_a c$ has meaning only when c can be produced by raising the number a to some power which is a number already defined. In the phrase "has only meaning for the present" is contained a possibility of new extensions of the domain of number. But before we pass to those extensions we shall first make use of the idea of evolution just defined to extend the notion of power also to cases in which the exponent is a fractional number.

According to the original definition of involution, a^b was meaningless except where b was a result of counting. But afterwards, even powers which had for their exponents zero or a negative integer could be invested with meaning. Now we have to consider the arithmetical combination "a raised to the fractional power $\frac{p}{q}$." The principle of no exception compels us to give to the arithmetical combination "a to the $\frac{p}{q}^{th}$ power" a significance such that all the rules of computation will hold with respect to it. Now, one rule that holds is, that the m^{th} power of the n^{th} power of a is equal to the $(m \times n)^{th}$ power of a. Consequently, the q^{th} power of a raised to the $\frac{p}{q}^{th}$ power must be equal to a raised to a power whose exponent is equal to $\frac{p}{q}$ times q. But the last-mentioned product gives, according to the definition of division, the number p. Consequently the sym-

bol a to the $\frac{p}{q}^{th}$ power is so constituted that its q^{th} power is equal to a to the p^{th} power, that is, is equal to the q^{th} root of a^p. Similarly, we find that the symbol "a to the minus $\frac{p}{q}^{th}$ power" must be put equal to 1 divided by the q^{th} root of a to the p^{th} power, if we are to reckon with this symbol as we do with real powers. Again, just as a to the b^{th} power is invested with meaning when b is a fractional number, so some meaning harmonious with the principle of no exception must be imparted to the b^{th} root of c where b is a positive or negative fractional number. For example, the three-fourthsth root of 8 is equal to 8 to the $\frac{4}{3}$ power, that is, to the cube root of 8 to the 4^{th} power, or 16.

The principle underlying arithmetic now also compels us to give to the symbol the "b^{th} root of c" a meaning when c is not the b^{th} power of any number yet defined. First, let c be any *positive* integer or fraction. Then always to be able to reckon with the b^{th} root of c in the same way that we do with extractible roots, we must agree always to put the b^{th} power of the b^{th} root of c equal to c—for example, $(\sqrt[2]{3})^2$ always exactly equal to 3. A careful inspection of the new symbols, which we will also call numbers, shows, that though no one of them is exactly equal to a number hitherto defined, yet by a certain extension of the notions greater and less, two numbers of the character of numbers already defined may be found for each such new number, such that the new number is greater than the one and less than the other of the two, and that further, these two numbers may be made to differ from each other by as small a quantity as we please. For example,

$$(\tfrac{7}{5})^3 = \tfrac{343}{125} = 2\tfrac{93}{125} < 3 < 3\tfrac{3}{8} = \tfrac{27}{8} = (\tfrac{3}{2})^3.$$

The number 3, as we see, is here included between two limits which are the third powers of two numbers $\frac{7}{5}$ and $\frac{3}{2}$ whose difference is $\frac{1}{10}$. We could also have arranged it so that the difference should be equal to $\frac{1}{100}$, or to any specified number, however small. Now, instead of putting the symbol "less than" between $(\frac{7}{5})^3$ and 3, and between 3 and $(\frac{3}{2})^3$, let us put it between their third roots; for example, let us say:

$$\tfrac{7}{5} < \sqrt[3]{3} < \tfrac{3}{2}, \text{ meaning by this that } (\tfrac{7}{5})^3 < 3 < (\tfrac{3}{2})^3.$$

In this sense we may say that the new numbers always lie *between*

two old numbers whose difference may be made as small as we please. Numbers possessing this property are called *irrational* numbers, in contradistinction to the numbers hitherto defined, which are termed *rational.* The considerations which before led us to negative rational numbers, now also lead us to negative irrational numbers. The repeated application of addition and multiplication as of their inverse processes to irrational numbers, (numbers which though not exactly equal to previously defined rational numbers may yet be brought as near to them as we please,) again simply leads to numbers of the same class.

A totally new domain of numbers is reached, however, when we attempt to impart meaning to *the square roots of negative numbers.* The square root of minus 9 is neither equal to plus 3 nor to minus 3, since each multiplied by itself gives plus 9, nor is it equal to any other number hitherto defined. Accordingly, the square root of minus 9 is a new number-form, to which, harmoniously with the principle of no exception, we may give the definition that $(\sqrt[2]{-9})^2$ shall always be put equal to minus 9.* Keeping to this definition we see at once that $\sqrt{-a}$, where a is any positive rational or irrational number, is a symbol which can be put equal to the product of $\sqrt{+a}$ with $\sqrt{-1}$. In extending to these new numbers the rights of arithmetical citizenship, in calling them also "numbers," and so shaping their definition that we can reckon with them by the same rules as with already defined numbers, we obtain a fourth extension of the domain of numbers which has become of the greatest importance for the progress of all branches of mathematics. The newly defined numbers are called *imaginary*, in contradistinction to all heretofore defined, which are called *real.* Since all imaginary numbers can be represented as products of real numbers with the square root of minus one, it is convenient to introduce for this one imaginary num ber some concise symbol. This symbol is the first letter of the word imaginary, namely, i; so that we can always put for such an expression as $\sqrt{-9}$, $3 . i$.

If we combine real and imaginary numbers by operations of the

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* Henceforward we shall use the simpler sign $\sqrt{\ }$ for $\sqrt[2]{\ }$.

first and second degree, always supposing that we follow in our reckoning with imaginary numbers the same rules that we do in reckoning with real numbers, we always arrive again at real or imaginary numbers, excepting when we join together a real and an imaginary number by addition or its inverse operations. In this case *we reach the symbol* $a + i \,.\, b$, where a and b stand for real numbers. Agreeably to the principle of no exception we are permitted to reckon with $a + ib$ according to the same rules of computation as with symbols previously defined, if for the second power of i we always substitute minus 1.

In the numerical combination $a + ib$, which we also call number, we have found the most general numerical form to which the laws of arithmetic can lead, even though we wished to extend the limits of arithmetic still further. Of course, we must represent to ourselves here by a and b either zero or positive or negative rational or irrational numbers. If b is zero, $a + ib$ represents all real numbers; if a is zero, it stands for all purely imaginary numbers. This general number $a + ib$ is called a *complex number*, so that the complex number includes in itself as special cases all numbers heretofore defined. By the introduction of irrational, purely imaginary, and the still more general complex numbers, all combinations become invested with meaning which the operations of the third degree can produce. For example, the fifth root of 5 is an irrational number, the logarithm of 2 to the base 10 is an irrational number. The logarithm of minus 1 to the base 2 is a purely imaginary number; the fourth root of minus 1 is a complex number. Indeed, we may recognise, proceeding still further, *that every combination of two complex numbers, by means of any of the operations of the first, second, or third degree will lead in turn to a complex number*, that is to say, never furnishes occasion, by application of the principle of no exception, for inventing new forms of numbers.

A certain limit is thus reached in the construction of arithmetic. But such a limit was also twice previously reached. After the investigation of addition and its inverse operations, we reached no other numbers except zero and positive and negative whole numbers, and every combination of such numbers by operations of the

first degree led to no new numbers. After the investigation of multiplication and its inverse operations, the positive or negative fractional numbers and "infinitely great" were added, and again we could say that the combination of two already defined numbers by operations of the first and second degree in turn also always led to numbers already defined. Now we have reached a point at which we can say that the combination of two complex numbers by all operations of the first, second, and third degree must again always lead to complex numbers; only that now such a combination does not necessarily always lead to a single number, but may lead to many regularly arranged numbers. For example, the combination "logarithm of minus one to a positive base" furnishes a countless number of results which form an arithmetical series of purely imaginary numbers. *Still, in no case now do we arrive at new classes of numbers.* But just as before the ascent from multiplication to involution brought in its train the definition of new numbers, so it is also possible that *some new operation springing out of involution as involution sprang from multiplication might furnish the germ of other new numbers which are not reducible to* $a + ib$. As a matter of fact, mathematicians have asked themselves this question and investigated the direct operation of the fourth degree, together with its inverse processes. The result of their investigations was, that an operation which springs from involution as involution sprang from multiplication is incapable of performing any real mathematical service; the reason of which is, that in involution the laws of commutation and association do not hold. It also further appeared that the operations of the fourth degree could not give rise to new numbers. No more so can operations of still higher degrees. With respect to quaternions, which many might be disposed to regard as new numbers, it will be evident that though quaternions are valuable means of investigation in geometry and mechanics they are not numbers of arithmetic, because the rules of arithmetic are not unconditionally applicable to them.

The building up of arithmetic is thus completed. The extensions of the domain of number are ended. It only remains to be asked why the science of arithmetic appears in its structure so logi-

cal, natural, and unarbitrary; why zero, negative, and fractional numbers appear as much derived and as little original as irrational, imaginary, and complex numbers? We answer, wholly and alone in virtue of the logical application of the monistic principle of arithmetic, the principle of no exception.

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